

ON A CONJECTURE OF SCHMIDT FOR THE PARAMETRIC GEOMETRY OF NUMBERS

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ABSTRACT. With the help of the recently introduced parametric geometry of numbers by W. M. Schmidt and L. Summerer, we prove a strong version of a conjecture of Schmidt concerning the successive minima of a lattice.

1. INTRODUCTION

Among the conjectures proposed by W. M. Schmidt in 1983, one is concerned with the parametric geometry of numbers [4, Conjecture 2]. This conjecture was proven in 2012 by N. G. Moshchevitin [1, Theorem 1]. The goal of this paper is to prove a stronger statement along the same lines and we will show that this generalization is the best possible. We start by recalling Moshchevitin's result, using the notations of D. Roy in [2].

Fix an integer $n \geq 2$. For each non-zero $\xi \in \mathbb{R}^{n+1}$, we associate the family of convex bodies

$$\mathcal{C}_\xi(Q) := \{\mathbf{x} \in \mathbb{R}^{n+1}; \|\mathbf{x}\| \leq 1, |\mathbf{x} \cdot \xi| \leq Q^{-1}\} \quad (Q \geq 1),$$

where $\mathbf{x} \cdot \mathbf{y}$ denotes the standard scalar product in \mathbb{R}^n and $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ denotes the euclidean norm of \mathbf{x} . Define

$$L_{\xi,j}(q) = \log \lambda_j(\mathcal{C}_\xi(e^q); \mathbb{Z}^{n+1}) \quad (q \geq 0; 1 \leq j \leq n+1),$$

where $\lambda_j(\mathcal{C}; \Lambda)$ is defined for a convex body \mathcal{C} and lattice Λ in \mathbb{R}^{n+1} to be the j -th minimum of \mathcal{C} with respect to Λ , i.e. the smallest $\lambda \geq 0$ such that $\lambda\mathcal{C}$ contains at least j linearly independent elements of Λ . Clearly, we have

$$L_{\xi,1}(q) \leq \cdots \leq L_{\xi,n+1}(q) \quad (q \geq 0).$$

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The functions $L_{\xi,j} : [0, \infty) \longrightarrow \mathbb{R}$ ($1 \leq j \leq n+1$) are continuous and piecewise linear, with slopes alternating between 0 and 1 (see [2, §2], [6, §3]). Moreover, since the volume of $\mathcal{C}_\xi(e^q)$ is bounded below and above by multiples of e^{-q} , Minkowski's theorem implies that

$$q - \sum_{j=1}^{n+1} L_{\xi,j}(q)$$

is a bounded function in q , and so the average of the $L_{\xi,j}$'s is $q/(n+1)$. If the coordinates of ξ are linearly independent over \mathbb{Q} , then for each $j = 1, \dots, n+1$, there exists arbitrarily large values of q such that

$$L_{\xi,j}(q) = L_{\xi,j+1}(q)$$

(see [5, Theorem 1]). On the other hand, we have the following result.

Theorem 1 (N. G. Moshchevitin, 2012). *For each integer k with $2 \leq k \leq n$, there exists $\xi \in \mathbb{R}^{n+1}$ whose coordinates are linearly independent over \mathbb{Q} such that*

$$\lim_{q \rightarrow \infty} \left(L_{\xi,k-1}(q) - \frac{q}{n+1} \right) = -\infty \quad \text{and} \quad \lim_{q \rightarrow \infty} \left(L_{\xi,k+1}(q) - \frac{q}{n+1} \right) = \infty.$$

Thus, the functions $L_{\xi,k-1}(q)$ and $L_{\xi,k+1}(q)$ can diverge from each other on each side by $q/(n+1)$. Our main result improves upon these estimates, and is stated as follows.

Theorem 2. *For each integer k with $2 \leq k \leq n$, there exist uncountably many vectors $\xi \in \mathbb{R}^{n+1}$ whose coordinates are linearly independent over \mathbb{Q} such that*

$$\lim_{q \rightarrow \infty} \frac{L_{\xi,k-1}(q)}{q} = 0 \quad \text{and} \quad \liminf_{q \rightarrow \infty} \frac{L_{\xi,k+1}(q)}{q} = \frac{1}{n-k+2}.$$

Further, this result is the best possible in the following sense.

Theorem 3. *Let k be an integer with $2 \leq k \leq n$, and suppose that ξ is a point in \mathbb{R}^{n+1} whose coordinates are linearly independent over \mathbb{Q} and which satisfies*

$$\lim_{q \rightarrow \infty} \frac{L_{\xi,k-1}(q)}{q} = 0.$$

Then, we have

$$\liminf_{q \rightarrow \infty} \frac{L_{\xi,k+1}(q)}{q} \leq \frac{1}{n-k+2}.$$

In the following section, we state Schmidt's original conjecture, and we justify the above reformulation of Moshchevitin's result. In section 3, we use the results of [3, §4] to prove Theorem 2. Finally, section 4 provides a proof of theorem 3 by using Schmidt and Summerer's parametric geometry of numbers.

2. LINK WITH SCHMIDT'S ORIGINAL CONJECTURE

For each $N \in \mathbb{R}$ with $N \geq 1$ and for each $\xi = (1, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, Schmidt [4] introduced the lattice $\Lambda(\xi, N) \subset \mathbb{R}^{n+1}$ generated by the vectors

$$\mathbf{v}_0 = (N^{-1}, N^{1/n}\xi_1, \dots, N^{1/n}\xi_n), \quad \mathbf{v}_1 = (0, -N^{1/n}, \dots, 0), \quad \dots, \quad \mathbf{v}_n = (0, 0, \dots, -N^{1/n}),$$

and defined

$$\mu_j(\xi, N) = \lambda_j(\mathcal{B}; \Lambda(\xi, N)) \quad (1 \leq j \leq n+1)$$

where $\mathcal{B} = \{(y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}; |y_i| \leq 1, i = 0, \dots, n\}$ is the unit hypercube in \mathbb{R}^{n+1} .

With these notations, he conjectured the following result, later proven by Moshchevitin.

Theorem 4 (N. G. Moshchevitin, 2012). *Let k be an integer with $2 \leq k \leq n$. There exists real numbers $\xi_1, \dots, \xi_n \in [0, 1)$ such that*

- $1, \xi_1, \dots, \xi_n$ are linearly independent over \mathbb{Q} ;
- $\lim_{N \rightarrow \infty} \mu_{k-1}(\xi, N) = 0$ and $\lim_{N \rightarrow \infty} \mu_{k+1}(\xi, N) = \infty$, where $\xi = (1, \xi_1, \dots, \xi_n)$.

In fact, Schmidt's original conjecture omits the linear independence condition, but as Moshchevitin mentions in his article, (see [1, §3]), the conjecture is trivial without this hypothesis.

To show the equivalence between Theorems 1 and 4, fix a point $\xi = (1, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$ whose coordinates are linearly independent over \mathbb{Q} , and fix an integer k with $2 \leq k \leq n$. In [1, §1], Moshchevitin begins by observing that

$$\mu_j(\xi, N) = \lambda_j(\mathcal{K}_\xi(N); \mathbb{Z}^{n+1}) \quad (N \geq 1, 1 \leq j \leq n+1),$$

where

$$\mathcal{K}_\xi(N) = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1}; |x_0| \leq N, |x_0\xi_j - x_j| \leq N^{-1/n}, j = 1, \dots, n\}.$$

Consequently, the second statement of theorem 4 can be rewritten as

$$(1) \quad \lim_{N \rightarrow \infty} \lambda_{k-1}(\mathcal{K}_\xi(N); \mathbb{Z}^{n+1}) = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} \lambda_{k+1}(\mathcal{K}_\xi(N); \mathbb{Z}^{n+1}) = \infty.$$

Meanwhile, Mahler's duality theorem yields

$$\lambda_j(\mathcal{K}_\xi(N); \mathbb{Z}^{n+1}) \lambda_{n-j+2}(\mathcal{K}_\xi^*(N); \mathbb{Z}^{n+1}) \asymp 1 \quad (1 \leq j \leq n+1),$$

where

$$\mathcal{K}_\xi^*(N) = \{\mathbf{x} \in \mathbb{R}^{n+1}; |\mathbf{x} \cdot \xi| \leq N^{-1}, \|\mathbf{x}\| \leq N^{1/n}\}$$

is essentially the convex body dual to $\mathcal{K}_\xi(N)$. Thus, the conditions in (1) become

$$(2) \quad \lim_{N \rightarrow \infty} \lambda_{n+3-k}(\mathcal{K}_\xi^*(N); \mathbb{Z}^{n+1}) = \infty \quad \text{and} \quad \lim_{N \rightarrow \infty} \lambda_{n+1-k}(\mathcal{K}_\xi^*(N); \mathbb{Z}^{n+1}) = 0.$$

On the other hand, since $\mathcal{C}_\xi(e^q) = e^{-q/(n+1)} \mathcal{K}_\xi^*(e^{nq/(n+1)})$, it follows that

$$L_{\xi,j}(q) = \frac{q}{n+1} + \log \lambda_j(\mathcal{K}_\xi^*(e^{nq/(n+1)}); \mathbb{Z}^{n+1}) \quad (1 \leq j \leq n+1).$$

Thus, the conditions in (2) can be rewritten as

$$\lim_{q \rightarrow \infty} \left(L_{\xi,n+3-k}(q) - \frac{q}{n+1} \right) = \infty \quad \text{and} \quad \lim_{q \rightarrow \infty} \left(L_{\xi,n+1-k}(q) - \frac{q}{n+1} \right) = -\infty.$$

The equivalence between theorems 1 and 4 follows.

3. PROOF OF THE MAIN RESULT

In order to prove Theorem 2, we need to establish some preliminary results which rely on the following basic construction.

Proposition 1. *Let $a, b, c, \alpha, \beta, \gamma \in (0, \infty)$ with $a < b < c$. There exists a unique choice of real numbers $r, s, t, u \in (0, \infty)$ with $r < s < t < u$ and a unique triplet of continuous and piecewise linear functions (A, B, C) on $[r, u]$ such that the union of their graphs is as in Figure 1, i.e.*

i) for all $q \in [r, u]$, we have

$$(3) \quad A(q) \leq B(q) \leq C(q) \quad \text{and} \quad \frac{1}{\alpha}A(q) + \frac{1}{\beta}B(q) + \frac{1}{\gamma}C(q) = q;$$

ii) the function A is constant equal to a on $[r, t]$, has slope α on $[t, u]$, and satisfies $A(u) = b$;

iii) the function B has slope β on $[r, s]$, is constant equal to b on $[s, u]$, and satisfies $B(r) = a$;

iv) the function C is constant equal to b on $[r, s]$, has slope γ on $[s, t]$, and is constant equal to c on $[t, u]$.

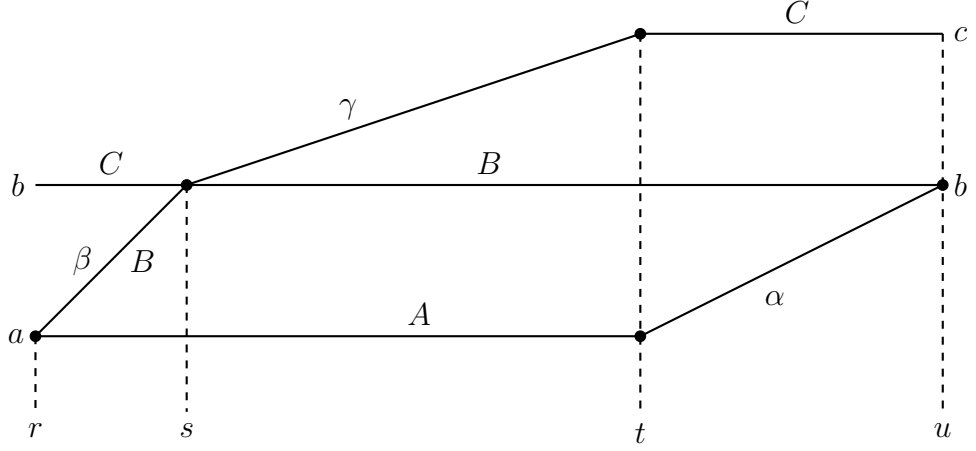


FIGURE 1.

Proof. If there exist real numbers r, s, t, u and functions A, B, C as in the claim, then substituting q by r, s, t, u in the second condition of (3) yields, respectively,

$$(4) \quad r = \frac{a}{\alpha} + \frac{a}{\beta} + \frac{b}{\gamma}; \quad s = \frac{a}{\alpha} + \frac{b}{\beta} + \frac{b}{\gamma}; \quad t = \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}; \quad u = \frac{b}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma},$$

which uniquely determines them all.

Now, let r, s, t, u be given by (4). Since $r < s < t < u$, there exists a unique triplet of continuous functions (A, B, C) on $[r, u]$ with constant slopes on $[r, s]$, $[s, t]$ and $[t, u]$, and with

$$A(r) = A(s) = A(t) = a \quad \text{and} \quad A(u) = b,$$

$$B(r) = a \quad \text{and} \quad B(s) = B(t) = B(u) = b,$$

$$C(r) = C(s) = b \quad \text{and} \quad C(t) = C(u) = c.$$

Thus, the function $F = \frac{1}{\alpha}A + \frac{1}{\beta}B + \frac{1}{\gamma}C$ is continuous and of constant slope on each of the interval $[r, s]$, $[s, t]$, and $[t, u]$. By construction, we have that $F(q) = q$ for $q = r, s, t, u$. Thus,

$$F(q) = q \quad \text{for all } q \in [r, u].$$

Since A and C are constant on $[r, s]$, this implies that B has slope β on $[r, s]$. Similarly, we deduce that C has slope γ on $[s, t]$, and that A has slope α on $[t, u]$. \square

Proposition 2. *With the same notation as above, suppose that $b/a < c/b$. Then, we have*

$$(5) \quad \max_{q \in [r, u]} \frac{A(q)}{q} = \frac{a}{r} \quad \text{and} \quad \min_{q \in [r, u]} \frac{C(q)}{q} = \frac{b}{s}.$$

Proof. First, using (4) note that

$$\frac{a}{t} < \frac{b}{u} < \frac{a}{r} \quad \text{and} \quad \frac{b}{s} < \frac{b}{r} < \frac{c}{u} < \frac{c}{t}.$$

Since $a/r < \alpha$ and $b/s < \gamma$, it follows that the ratio $A(q)/q$ is decreasing on $[r, t]$ and increasing on $[t, u]$, and that the ratio $C(q)/q$ is decreasing on $[r, s]$, increasing on $[s, t]$ and decreasing on $[t, u]$. The conclusion follows straightforwardly. \square

Let Δ denote the set of sequences $(a_m)_{m \in \mathbb{Z}}$ of positive reals which satisfy

$$1 < \frac{a_{m+1}}{a_m} < \frac{a_{m+2}}{a_{m+1}} \quad (m \in \mathbb{Z}),$$

$$\lim_{m \rightarrow -\infty} a_m = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \frac{a_{m+1}}{a_m} = +\infty.$$

The following result further extends the preceding propositions.

Proposition 3. *Let $(a_m)_{m \in \mathbb{Z}} \in \Delta$ and let $\alpha, \beta, \gamma \in (0, \infty)$. Define*

$$(6) \quad r_m = \frac{a_m}{\alpha} + \frac{a_m}{\beta} + \frac{a_{m+1}}{\gamma} \quad (m \in \mathbb{Z}).$$

Then, there exists a unique triplet of continuous and piecewise linear functions (A, B, C) on $(0, \infty)$ whose restriction to the interval $[r_m, r_{m+1}]$ fulfills the conditions of Proposition 1 with $a = a_m$, $b = a_{m+1}$ and $c = a_{m+2}$ for each $m \in \mathbb{Z}$. Moreover, we have

$$(7) \quad \lim_{q \rightarrow \infty} A(q) = \infty, \quad \limsup_{q \rightarrow \infty} \frac{A(q)}{q} = 0 \quad \text{and} \quad \liminf_{q \rightarrow \infty} \frac{C(q)}{q} = \frac{\beta\gamma}{\beta + \gamma}.$$

Proof. Let $(a_m)_{m \in \mathbb{Z}} \in \Delta$, and define

$$(8) \quad s_m = \frac{a_m}{\alpha} + \frac{a_{m+1}}{\beta} + \frac{a_{m+1}}{\gamma} \quad \text{and} \quad t_m = \frac{a_m}{\alpha} + \frac{a_{m+1}}{\beta} + \frac{a_{m+2}}{\gamma} \quad (m \in \mathbb{Z}).$$

By setting $a = a_m$, $b = a_{m+1}$ and $c = a_{m+2}$, Proposition 1 and (4) yield for each $m \in \mathbb{Z}$ a triplet of continuous and piecewise linear functions $(A^{(m)}, B^{(m)}, C^{(m)})$ on $[r, u] = [r_m, r_{m+1}]$.

Since the triplets $(A^{(m-1)}, B^{(m-1)}, C^{(m-1)})$ and $(A^{(m)}, B^{(m)}, C^{(m)})$ coincide at the point r_m and are equal to (a_m, a_m, a_{m+1}) for each $m \in \mathbb{Z}$, it follows that the sequence of triplets

of functions $(A^{(m)}, B^{(m)}, C^{(m)})$ with $m \in \mathbb{Z}$ determine a unique triplet of continuous and piecewise linear functions (A, B, C) on $\bigcup_{m \in \mathbb{Z}} [r_m, r_{m+1}] = (0, \infty)$. Now, Proposition 2 gives

$$(9) \quad \max_{q \in [r_m, r_{m+1}]} \frac{A(q)}{q} = \frac{a_m}{r_m} \quad \text{and} \quad \min_{q \in [r_m, r_{m+1}]} \frac{C(q)}{q} = \frac{a_{m+1}}{s_m},$$

and so

$$\limsup_{q \rightarrow \infty} \frac{A(q)}{q} = \lim_{m \rightarrow \infty} \frac{a_m}{r_m} = 0 \quad \text{and} \quad \liminf_{q \rightarrow \infty} \frac{C(q)}{q} = \lim_{m \rightarrow \infty} \frac{a_{m+1}}{s_m} = \frac{\beta\gamma}{\beta + \gamma}.$$

□

Our next result uses the notion of *generalized* $(n+1)$ -system introduced by D. Roy in [3]. It provides a good approximation of the functions \mathbf{L}_ξ for non-zero point $\xi \in \mathbb{R}^{n+1}$ (see [3] for more details). We recall here the definition .

Definition. Let I be a subinterval of $[0, \infty)$ with non-empty interior. A *generalized* $(n+1)$ -system on I is a map $\mathbf{P} = (P_1, \dots, P_{n+1}) : I \longrightarrow \mathbb{R}^{n+1}$ with the following properties.

- (G1) For each $q \in I$, we have $0 \leq P_1(q) \leq \dots \leq P_{n+1}(q)$ and $P_1(q) + \dots + P_{n+1}(q) = q$.
- (G2) If H is a non-empty open subinterval of I on which \mathbf{P} is differentiable, then there are integers \underline{r}, \bar{r} with $1 \leq \underline{r} \leq \bar{r} \leq n+1$ such that $P_{\underline{r}}, P_{\underline{r}+1}, \dots, P_{\bar{r}}$ coincide on the whole interval H and have slope $1/(\bar{r} - \underline{r} + 1)$ while any other component P_j of P is constant on H .
- (G3) If q is an interior point of I at which \mathbf{P} is not differentiable, if $\underline{r}, \bar{r}, \underline{s}, \bar{s}$ are the integers for which

$$(10) \quad P'_j(q^-) = \frac{1}{\bar{r} - \underline{r} + 1} \quad (\underline{r} \leq j \leq \bar{r}) \quad \text{et} \quad P'_j(q^+) = \frac{1}{\bar{s} - \underline{s} + 1} \quad (\underline{s} \leq j \leq \bar{s})$$

and if $\underline{r} \leq \bar{s}$, then we have $P_{\underline{r}}(q) = P_{\underline{r}+1}(q) = \dots = P_{\bar{s}}(q)$.

We now combine the previous Propositions to establish the following result.

Proposition 4. Let k be an integer with $2 \leq k \leq n$. With the notation of Proposition 3, suppose that $\alpha = 1/(k-1)$, $\beta = 1$ and $\gamma = 1/(n+1-k)$. For all $q > 0$, let

$$P_1(q) = \dots = P_{k-1}(q) = A(q), \quad P_k(q) = B(q) \quad \text{and} \quad P_{k+1}(q) = \dots = P_{n+1}(q).$$

Then the function $\mathbf{P} : (0, \infty) \longrightarrow \mathbb{R}^{n+1}$ defined by

$$\mathbf{P}(q) := (P_1(q), \dots, P_{n+1}(q)) \quad (q > 0)$$

is an generalized $(n + 1)$ -system on $(0, \infty)$. Moreover, we have

$$\lim_{q \rightarrow \infty} P_1(q) = \infty, \quad \limsup_{q \rightarrow \infty} \frac{P_{k-1}(q)}{q} = 0 \quad \text{and} \quad \liminf_{q \rightarrow \infty} \frac{P_{k+1}(q)}{q} = \frac{1}{n - k + 2}.$$

Proof. The components P_1, \dots, P_{n+1} of \mathbf{P} are continuous and piecewise linear on $(0, \infty)$. They satisfy

$$0 \leq P_1(q) \leq \dots \leq P_{n+1}(q) \quad \text{and} \quad P_1(q) + \dots + P_{n+1}(q) = q \quad (q > 0).$$

The function \mathbf{P} is differentiable on $(0, \infty)$ except at the points r_m, s_m, t_m given by (6) and (8). On each of the interval $[r_m, s_m]$, $[s_m, t_m]$, $[t_m, r_{m+1}]$, the components P_1, \dots, P_{n+1} are constant except for few, say h of them, which coincide on the interval and which have slope $1/h$. At the point r_m , the slopes of P_1, \dots, P_{k-1} go from $1/(k-1)$ to 0, while the slope of P_k goes from 0 to 1, and all these functions take the same value, i.e.

$$P_1(r_m) = \dots = P_k(r_m) \quad (m \in \mathbb{Z}).$$

At the point s_m , the function P_k goes from slope 1 to slope 0, while the slopes of P_{k+1}, \dots, P_{n+1} go from 0 to $1/(n-k+1)$, and similary.

$$P_k(s_m) = P_{k+1}(s_m) = \dots = P_{n+1}(s_m) \quad (m \in \mathbb{Z}).$$

Finally, at the point t_m , the slopes of P_{k+1}, \dots, P_{n+1} go from $1/(n-k+1)$ to 0, while the slopes of P_1, \dots, P_{k-1} go from 0 to $1/(k-1)$, and we have

$$P_1(t_m) = \dots = P_{k-1}(t_m) < P_k(t_m) < P_{k+1}(t_m) = \dots = P_n(t_m) \quad (m \in \mathbb{Z}).$$

Therefore, the function \mathbf{P} is an *generalized $(n + 1)$ -system* on $(0, \infty)$. The second assertion of the proposition follows from (7). \square

In [3, §4], D. Roy shows that for each *generalized $(n + 1)$ -system* \mathbf{P} on $[q_0, \infty)$ with $q_0 \geq 0$, there exists a non-zero point ξ of \mathbb{R}^{n+1} such that the difference $\mathbf{L}_\xi - \mathbf{P}$ is bounded. Then, we have

$$\limsup_{q \rightarrow \infty} \frac{L_{\xi,j}(q)}{q} = \limsup_{q \rightarrow \infty} \frac{P_j(q)}{q} \quad \text{and} \quad \liminf_{q \rightarrow \infty} \frac{L_{\xi,j}(q)}{q} = \liminf_{q \rightarrow \infty} \frac{P_j(q)}{q} \quad (1 \leq j \leq n+1).$$

In the context of Proposition 4, this guarantees the existence of a point $\xi \in \mathbb{R}^{n+1}$ with

$$(11) \quad \limsup_{q \rightarrow \infty} \frac{L_{\xi,k-1}(q)}{q} = 0 \quad \text{and} \quad \liminf_{q \rightarrow \infty} \frac{L_{\xi,k+1}(q)}{q} = \frac{1}{n - k + 2}.$$

Moreover, since $\lim_{q \rightarrow \infty} P_1(q) = \infty$, the function $L_{\xi,1}$ is unbounded. It follows that ξ is a point whose coordinates are linearly independent over \mathbb{Q} .

To finish the proof, it remains to show that one can construct uncountably many such points. For each $\theta \in (0, \infty)$, we define

$$a_m^{(\theta)} = \theta 2^{m^3} \quad (m \in \mathbb{Z}).$$

Then, the sequence $(a_m^{(\theta)})_{m \in \mathbb{Z}}$ belongs to Δ , and Propositions 3 and 4 associate to it an *generalized* $(n+1)$ -system $\mathbf{P}^{(\theta)}$ on $(0, \infty)$, and a point $\xi^{(\theta)} \in \mathbb{R}^{n+1}$. Extending the notation in an obvious manner gives

$$\begin{aligned} r_m^{(\theta)} &= k a_m^{(\theta)} + (n - k + 1) a_{m+1}^{(\theta)} < (n + 1) a_{m+1}^{(\theta)} \\ t_m^{(\theta)} &= (k - 1) a_m^{(\theta)} + a_{m+1}^{(\theta)} + (n - k + 1) a_{m+2}^{(\theta)} > a_{m+2}^{(\theta)} \end{aligned}$$

for all $m \in \mathbb{Z}$, and $t_m^{(\theta)}/r_m^{(\theta)}$ tends to infinity with m . Thus, if $\theta, \theta' \in (0, \infty)$ with $\theta < \theta'$, then

$$r_m^{(\theta)} < r_m^{(\theta')} = (\theta'/\theta) r_m^{(\theta)} < t_m^{(\theta)},$$

for all sufficiently large $m \in \mathbb{Z}$, and so

$$\|\mathbf{P}^{(\theta')}(r_m^{(\theta')}) - \mathbf{P}^{(\theta)}(r_m^{(\theta')})\| \geq |P_1^{(\theta')}(r_m^{(\theta')}) - P_1^{(\theta)}(r_m^{(\theta')})| = |a_m^{(\theta')} - a_m^{(\theta)}| = (\theta' - \theta) 2^{m^3}.$$

This means that the difference $\mathbf{P}^{(\theta')} - \mathbf{P}^{(\theta)}$ is unbounded. Thus, the points $\xi^{(\theta')}$ and $\xi^{(\theta)}$ are distinct, and consequently, the map $\theta \mapsto \xi^{(\theta)}$ is injective on $(0, \infty)$. Its image is therefore uncountable.

4. PROOF OF THEOREM 3

Let ξ be a point in \mathbb{R}^{n+1} whose coordinates are linearly independent over \mathbb{Q} . On the model of Schmidt and Summerer in [5, §1], we define

$$\underline{\varphi}_j(\xi) = \liminf_{q \rightarrow \infty} \frac{L_{\xi,j}(q)}{q} \quad \text{and} \quad \overline{\varphi}_j(\xi) = \limsup_{q \rightarrow \infty} \frac{L_{\xi,j}(q)}{q} \quad (1 \leq j \leq n+1).$$

In [5, §1], Schmidt and Summerer show that these quantities satisfy

$$(12) \quad \underline{\varphi}_{j+1}(\xi) \leq \overline{\varphi}_j(\xi) \quad (1 \leq j \leq n).$$

Now, suppose that $\overline{\varphi}_{k-1}(\xi) = 0$ for some integer k with $2 \leq k \leq n$. Since $q - \sum_{j=1}^{n+1} L_{\xi,j}(q)$ is a bounded function in q on $(0, \infty)$, we have that

$$(n - k + 2)\overline{\varphi}_k(\xi) \leq \limsup_{q \rightarrow \infty} \frac{1}{q} \sum_{j=k}^{n+1} L_{\xi,j}(q) = \limsup_{q \rightarrow \infty} \frac{1}{q} \left(q - \sum_{j=1}^{k-1} L_{\xi,j}(q) \right) = 1,$$

and so $\overline{\varphi}_k(\xi) \leq 1/(n - k + 2)$. This yields $\underline{\varphi}_{k+1}(\xi) \leq 1/(n - k + 2)$.

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